A NOTE ON THE MEAN VALUE PROPERTY

BY

ADRIANO M. GARSIA

Introduction. In a recent paper(1) A. Friedman and W. Littman have considered the following problem. In n-dimensional space is given a mass distribution μ with compact support S_μ of total mass one. Then, a function μ defined in a domain D is said to satisfy the mean value property in D with respect to μ if

$$u(x) = \int u(x + ty)d\mu(y) \tag{1.1}$$

for all $x \in D$ and sufficiently small $t > 0$.

In the above mentioned paper the authors show that if μ is positive and S_μ is not flat, then all functions satisfying the mean value property with respect to μ are analytic. If μ in addition satisfies certain restrictions, then the functions satisfying the mean value property with respect to μ fill only a finite dimensional vector space of polynomials.

In particular the latter result is shown to hold if μ is positive, S_μ is not flat and contains only a finite number of points. It is also shown that when μ is positive and concentrates equal masses at N points, then the polynomials satisfying the mean value property with respect to μ are all of degree less than or equal to $N(N-1)/2$.

It develops that the last mentioned result holds under lesser conditions on the mass μ.

It will be the purpose of this note to show that the following theorem holds.

THEOREM. *If μ concentrates all its mass on a finite set of points $S = P_1 \cup P_2 \cup \cdots \cup P_N$ whose convex hull is not flat and if in addition, setting $\mu_i = \int_{(P_i)} d\mu$ we have*

$$\mu_{i_1} + \mu_{i_2} + \cdots + \mu_{i_k} \neq 0 \qquad \text{for all } i \leqq i_1 < i_2 < \cdots < i_k \leqq N$$

then the functions satisfying the mean value property with respect to μ fill only a space of polynomials of degree less than or equal to $N(N-1)/2$.

1.1. **An auxiliary result.** To establish the theorem we shall need the following

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(1) *Editorial note.* See *Functions satisfying the mean value property,* immediately preceding this paper.

Lemma. *Let $\sigma_p(y_1, y_2, \cdots, y_N)$ for $p=1, 2, \cdots, N$ be polynomials in the indeterminates $y_1, y_2, \cdots, y_N$ defined as follows*

$$\sigma_p(y_1, y_2, \cdots, y_N) = \sum_{1\le i_1<\cdots<i_p\le N} c_{i_1,i_2,\cdots,i_p} y_{i_1}y_{i_2}\cdots y_{i_p}. \tag{1.11}$$

Then if $c_{i_1,i_2,\cdots,i_p} \neq 0$ for all $1 \leqq i_1 < i_2 < \cdots < i_p \leqq N$, all monomials of the type

$$\alpha = y_1^{k_1} y_2^{k_2} \cdots y_N^{k_N}, \qquad k_1 + k_2 + \cdots + k_N > \frac{N(N-1)}{2}$$

belong to the ideal generated by $\sigma_1, \sigma_2, \cdots, \sigma_N$.

Proof. Let $K=(k_1, k_2, \cdots, k_N)$ denote an N-tuple of non-negative integers such that $k_1 \geqq k_2 \geqq \cdots \geqq k_N$. We shall set $d(K)=k_1+k_2+\cdots+k_N$. If α is a monomial in the y_i's, we shall write $\exp \alpha = K = (k_1, k_2, \cdots, k_N)$ if and only if

$$\alpha = y_{i_1}^{k_1} y_{i_2}^{k_2} \cdots y_{i_N}^{k_N}$$

where $(i_1, i_2, \cdots, i_N)$ is some permutation of $(1, 2, \cdots, N)$. The set of all K such that $d(K)=M$ (for a fixed M) will be ordered by setting

$$K = (k_1, k_2, \cdots, k_N) \prec H = (h_1, h_2, \cdots, h_N)$$

if and only if

$$k_1 = h_1, \cdots, k_\nu = h_\nu, \quad \text{but} \quad k_{\nu+1} < h_{\nu+1} \qquad \text{for some } 1 \leqq \nu \leqq N-1.$$

We shall also adopt the following convention: for two polynomials $A(y_1, y_2, \cdots, y_N)$ and $B(y_1, y_2, \cdots, y_N)$ the relation $A \equiv B$ is to mean that the polynomial $A-B$ belongs to the ideal generated by $\sigma_1, \sigma_2, \cdots, \sigma_N$.

Now, suppose that $H=(h_1, h_2, \cdots, h_N)$ is such an N-tuple that

(1) $d(H)=M$.

(2) For every monomial α the relations $d(\exp \alpha)=M$, $\exp \alpha \prec H$ imply $\alpha \equiv 0$.

(3) For some permutation $(j_1, j_2, \cdots, j_N)$ of $(1, 2, \cdots, N)$

$$\alpha_0 = y_{j_1}^{h_1} y_{j_2}^{h_2} \cdots y_{j_N}^{h_N} \not\equiv 0. \tag{1.12}$$

Then one of the following must hold: either

$$\text{(a)} \qquad h_\lambda > h_{\lambda+1} + 1 \qquad \text{for some } 1 \leqq \lambda \leqq N-1$$

or

$$\text{(b)} \qquad h_\lambda \leqq h_{\lambda+1} + 1 \qquad \text{for all } 1 \leqq \lambda \leqq N-1.$$

In the case that (a) holds we express $y_{j_1}y_{j_2}\cdots y_{j_\lambda}$ by means of (1.11) written for $p=\lambda$ and obtain

$$y_{j_1}y_{j_2}\cdots y_{j_\lambda} \equiv \frac{-1}{c_{j_1,j_2,\cdots,j_\lambda}} \sum^{(i_1,i_2,\cdots,i_\lambda)} c_{i_1,i_2,\cdots,i_\lambda} y_{i_1}y_{i_2}\cdots y_{i_\lambda}$$
$$= c(y_{j_1},y_{j_2},\cdots,y_{j_\lambda})$$

where the sum is to include all the terms in (1.11) with the exception of the term in $y_{j_1}y_{j_2}\cdots y_{j_\lambda}$. Hence we get

$$(1.13) \qquad \alpha_0 \equiv y_{j_1}^{h_1-1}\, y_{j_2}^{h_2-1} \cdots y_{j_\lambda}^{h_\lambda-1}\, y_{j_{\lambda+1}}^{h_{\lambda+1}} \cdots y_{j_N}^{h_N}\, C(y_{j_1}, y_{j_2}, \cdots, y_{j_\lambda}).$$

It is clear that the expression on the right-hand side of (1.13) when expanded will yield a linear combination of monomials of degree M. We claim that for each monomial α that will be produced we shall have $\exp \alpha < H$. In fact, each monomial appearing in $C(y_{j_1}, y_{j_2}, \cdots, y_{j_\lambda})$ contains at most $\lambda-1$ of the variables $y_{j_1}y_{j_2}\cdots y_{j_\lambda}$. Thus for each α appearing on the right-hand side of (1.13) $\exp\alpha$ will not have all the first λ components as large as $h_1, h_2, \cdots, h_\lambda$. For, every exponent h_i with $i \geqq \lambda+1$ even if increased by one unit, can never become as large as one of the $h_1, h_2, \cdots, h_\lambda$.

From these considerations, in view of the hypothesis (2) we obtain $\alpha_0 \equiv 0$, a contradiction. Thus only (b) is possible. However, (b) implies that

$$d(H) \leqq Nh_N + 1 + 2 + \cdots + N - 1.$$

Now, if $d(H) > 1+2+\cdots+N-1$, we must have $Nh_N \geqq 1$; but this (in view of (1.11) written for $p=N$) would again imply $\alpha_0 \equiv 0$. Thus the only hypothesis compatible with $\alpha_0 \not\equiv 0$ is $d(H) \leqq N(N-1)/2$.

To complete the proof of the lemma, we observe that for a given M either all monomials of degree M are congruent to zero or there is a set of exponents H such that the conditions (1), (2), and (3) are satisfied.

1.2. **Proof of the theorem.** Let the point P_i on which μ concentrates the mass μ_i have coordinates

$$(x_{i,1}, x_{i,2}, \cdots, x_{i,n}).$$

Denote by y_i and L_p the differential operators

$$(1.21) \qquad y_i = \sum_{j=1}^{n} x_{i,j} \frac{\partial}{\partial x_j} \qquad (i = 1, 2, \cdots, N)$$

and

$$(1.22) \qquad L_p = \sum_{i=1}^{N} \mu_i y_i^p \qquad (p = 1, 2, \cdots, \text{etc.}).$$

A. Friedman and W. Littman have shown that every function $u(x_1, x_2, \cdots, x_n)$ which satisfies the mean value property with respect to μ is a weak solution of the differential operators in (1.22).

There is a short and elementary proof of this result which proceeds as follows. We multiply the equation (1.1) by a C^∞ function $\phi(x)$ whose support is compact and contained in D and obtain

$$(1.23)\qquad \int u(x)\phi(x)dx = \int \phi(x)\left[\int u(x+ty)d\mu(y)\right]dx.$$

Changing the order of integration and making the substitution $x+ty=z$ we can write (1.23) in the form

$$(1.24)\qquad \int u(x)\phi(x)dx = \int u(z)\left[\int \phi(z-ty)d\mu(y)\right]dz.$$

We then observe that, since ϕ is C^∞ and has compact support, the right-hand side of (1.24) can be differentiated under the integral sign as many times as we please. Since the left-hand side is constant, all the derivatives of the right-hand side must vanish. In particular, (for $t=0$) we must have

$$(1.25)\qquad \sum_{k_1+k_2+\cdots+k_n=p} \int y_1^{k_1}y_2^{k_2}\cdots y_n^{k_n}d\mu \int u(z)\frac{\partial^p\phi}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_n^{k_n}}dz = 0$$

for $p=1, 2, \cdots$, etc. This implies the assertion(2).

Let now t be a dummy variable and note that

$$(1.26)\qquad \sum_{p=1}^{\infty} L_p t^p = \sum_{i=1}^{N}\frac{\mu_i y_i t}{1-y_i t} = t\,\frac{Q(t)}{P(t)},$$

where we have set

$$P(t) = (1-y_1t)(1-y_2t)\cdots(1-y_Nt)$$

and

$$Q(t) = \sum_{i=1}^{N}\mu_i y_i Q_i(t),\qquad Q_i(t) = \prod^{j\neq i}(1-y_jt).$$

We can write (1.26) in the form

$$P(t)\sum_{p=1}^{\infty} L_p t^{p-1} = Q(t).$$

Since $Q(t)$ is a polynomial of degree $N-1$ we deduce that each of its coeffi-

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(2) As was pointed out by the above mentioned authors, when μ is a positive distribution, the infinite system (1.25) is equivalent to the mean value property. For then, the equation one obtains from (1.25) for $p=2$ is elliptic and by Weyl's lemma μ must necessarily be analytic. Thus the right-hand side of (I.1) is analytic in t and it is a constant (in t) if and only if all its derivatives vanish for $t=0$.

cients belongs to the ideal generated by $L_1, L_2, \cdots, L_N$. To calculate these coefficients we write $Q_i(t)$ in the form

$$Q_i(t) = \sum_{\nu=0}^{N-1} (-1)^\nu \sigma_{i,\nu} t^\nu,$$

where $\sigma_{i,\nu}$ denotes the elementary symmetric function of degree ν in the $N-1$ variables $y_1, y_2, \cdots, y_{i-1}, y_{i+1}, \cdots, y_N$. We then have

$$Q(t) = \sum_{i=1}^{N} \mu_i y_i \sum_{\nu=0}^{N-1} (-1)^\nu \sigma_{i,\nu} t^\nu = \sum_{\nu=0}^{N-1} (-1)^\nu \overset{\mu}{\sigma}_{\nu+1} t^\nu,$$

where we have set

$$\overset{\mu}{\sigma}_{\nu+1} = \sum_{i=1}^{N} \mu_i y_i \sigma_{i,\nu}. \tag{1.27}$$

It can be readily seen that each σ_ν^μ is of the type

$$\overset{\mu}{\sigma}_\nu = \sum_{1 \leq i_1 < \cdots < i_\nu \leq N} c_{i_1, i_2, \cdots, i_\nu} y_{i_1} y_{i_2} \cdots y_{i_\nu}. \tag{1.28}$$

But the actual expressions for the coefficients $c_{i_1, i_2, \cdots, i_\nu}$ can also be obtained. In fact, because of the symmetry of the problem, we need only to calculate the coefficient of $y_1, y_2, \cdots, y_{\nu+1}$ in (1.27). However, a monomial such as $y_1, y_2, \cdots, y_{\nu+1}$ can only be produced by the first $\nu+1$ terms in the sum (1.27). On the other hand, $\sigma_{i,\nu}$ for $1 \leqq i \leqq \nu+1$ contains exactly once the monomial $y_1 y_2 \cdots y_{i-1} y_{i+1} \cdots y_{\nu+1}$. The coefficient of $y_1 y_2 \cdots y_{\nu+1}$ in (1.27) is therefore $\mu_1 + \mu_2 + \cdots + \mu_{\nu+1}$. Consequently we must also have

$$c_{i_1, i_2, \cdots, i_\nu} = \mu_{i_1} + \mu_{i_2} + \cdots + \mu_{i_\nu}.$$

In view of the hypothesis of the theorem the lemma can be applied to deduce that the function $u(x_1, x_2, \cdots, x_n)$ must alo be a weak solution of all differential operators of the type

$$\alpha = y_1^{k_1} y_2^{k_2} \cdots y_N^{k_N} \qquad k_1 + k_2 + \cdots + k_N > \frac{N(N-1)}{2} \cdot \tag{1.29}$$

The assumption that the support $S = P_1 \cup P_2 \cup \cdots \cup P_N$ is not flat assures that the differential operators

$$\beta = \frac{\partial^{k_1+k_2+\cdots+k_N}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_N^{k_N}} \qquad k_1 + k_2 + \cdots + k_N > \frac{N(N-1)}{2} \tag{1.210}$$

can be expressed as linear combinations of the differential operators in (1.28). We thus obtain that $u(x_1, x_2, \cdots, x_n)$ is a weak solution of all operators in

(1.210). In other words, $u\ (x_1, x_2, \cdots, x_n)$ is a polynomial in its arguments of degree not exceeding $N(N-1)/2$.

Perhaps we should add the remark that from (1.26) it can be deduced that the finite system

$$L_1u = L_2u = \cdots = L_Nu = 0 \tag{1.211}$$

is equivalent to the infinite system $L_pu=0$ ($p=1, 2, \cdots$, etc.). Therefore each polynomial u satisfying (1.211) will also satisfy the mean value property with respect to μ.

UNIVERSITY OF MINNESOTA,
MINNEAPOLIS, MINNESOTA